

Self-organized states in cellular automata: Exact solution

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The spatial structure, fluctuations as well as all state probabilities of self-organized (steady) states of cellular automata can be found (almost) exactly and *explicitly* from their Markovian dynamics. The method is shown on an example of a natural sand pile model with a gradient threshold. [S1063-651X(98)06211-4]

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Complicated dynamics of various discrete systems may naturally be modeled by cellular automata (CA) having rather simple iteration rules. In particular, CA models are useful to study traffic jams [1], granular material dynamics [2], and self-organization [3,4]. The transport properties of systems, especially in the self-organized (SO) critical regime, were extensively studied (cf. [4–8]). However, for many applications, one needs to know an entire structure of the SO states. These may include (i) the temperature profiles for the convection dominated thermoconduction and turbulent convection [9] (as in the convective zone of the sun and stars), (ii) stability and average profiles of granular materials (e.g., the sand pile profiles) and granular flows [2,3], (iii) equilibrium and steady state profiles of plasma pressure and temperature in fusion devices [8,10], etc. Despite its importance, the problem of spatial structure and characteristics of the SO steady states has received minor attention [7,11–13]. In this paper, we propose a method that is equally applicable to any CA provided its rules are rather simple. We illustrate it on the simplest and most popular example of a one-dimensional sand pile with a gradient stability criterion. Note that this model is the closest to a natural pile of sand created by random sprinkling of sand and with a known repose angle. We focus on calculating an average slope (the calculation is more transparent) though calculating other characteristics is trivial.

An interesting result obtained is that the SO profiles of a local slope are *nontrivial* even for this simplest case. They are typically flat (e.g., linear) throughout the pile while a narrow region (similar to the boundary layer) with rapidly increasing gradient always occurs near the top of the pile. This picture is very similar to that experimentally measured in a strongly turbulent convection of a passive scalar (temperature) with a mean gradient [9]. The region of the supercritical (unstable) gradient may form near the bottom of the pile when the noise is strong enough to maintain almost continuous sand flow (overlapped avalanches). This is in excellent agreement with direct numerical simulations [10].

Among many sand pile models, the Abelian model is the only one that was proved to be *analytically* exactly solvable

[12,14]. This model is, however, rather unnatural since its stability criterion is given in terms of a site *height*, not *slope* of the pile (which depends on neighboring sites too). Another tacit assumption is one of *weak noise*: no sand is added to the pile unless at least one unstable site presents (i.e., until the very last avalanche is gone). Note that “waves of toppling” [13], which are the main point of the whole analysis, are well defined in the *weak noise* limit, only. We show that both these restrictions may be relaxed: (i) a site stability criterion may depend on a site’s neighbors and (ii) sand may be added at every time step thus affecting an avalanche dynamics. We also show that in the weak noise limit, all state probabilities can be calculated exactly in a *closed form*, while for the strong noise, they can be found *almost* exactly, i.e., with any *a priori* given accuracy.

For a general N -dimensional sand pile automaton, the procedure is as follows. (1) Reformulate the model in the representation where the stability is *local* and defined by the state of a site alone. This can be done at the expense of introducing a nonlocality to the toppling and noise rules (i.e., they may depend on states of adjacent sites too). (2) Consider the dynamics of a single site. Since toppling rules have no intrinsic memory, however, it is Markovian. Construct an N -dimensional Markov hyperlattice (an analog of a Markov chain in one dimension) with the transition probabilities defined by the CA rules. All the transition probabilities that depend on states of other sites are, for now, free parameters. Introducing a generating function, one can then solve the problem for a single site. (3) Noting that all sites are identical, we relate the Markov transition probabilities for different sites. Boundary conditions then uniquely define their values and, thus, the SO state of the pile. Note that a mean-field-type closure is needed at step (3) only.

Model. The model we consider consists of $L+1$ spatial sites, numbered from $x=0$ to $x=L$. To each site x is assigned a variable $h(x)$, the height of the site. CA rules are applied to the pile at each timestep. Sand grains are added randomly to sites with probabilities $p_{\text{sand}}(x)$ increasing the height by one. When the site x is unstable, i.e., if the local slope [difference of heights of two neighboring sites $h(x) - h(x+1)$] exceeds some critical value $\Delta h_{\text{crit}}(x)$, N_f sand grains topple onto the neighboring site $x+1$ (local, limited model [4]). Note that $N_f > 1$ corresponds to the physical situation where friction between sand grains at rest is greater than friction of those in motion. Sand is expelled from the pile through the right end $x=L$.

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tive. Here the first is the normalization condition: $\sum p_j = 1$, and second relates $F(\zeta, t)$ to the first moment (e.g., expectation value) of the probability distribution. Higher moments (i.e., standard deviation, etc.) are obtained from higher derivatives of $F(\zeta, t)$. Multiplying each of Eqs. (2) for \dot{p}_k respectively by ζ^k and taking sum over all $0 \leq k < \infty$, we straightforwardly obtain an equation for a generating function $F = \Phi + \Psi$. Since we are interested only in a steady state, we set $\dot{F} = 0$. Then it reads

$$F(\zeta)\{\bullet\} + \Psi(\zeta)(\zeta^{-2N_f} - 1)[\alpha^{-1} + \{\bullet\}] = p_0(\zeta^{-1} - 1), \quad (5)$$

where $\alpha = \beta$, $\epsilon = \epsilon^*/\alpha$, $\delta = \delta^*/\alpha$ and $\{\bullet\} = (\zeta + \zeta^{-1} - 2)(1 + \epsilon\alpha\zeta^{N_f} + \delta\alpha\zeta^{2N_f}) + \epsilon(\zeta^{N_f} - 1) + \delta(\zeta^{2N_f} - 1)$. Here $\Phi(\zeta) = \sum_{k=0}^{Z_c} \zeta^k p_k$, $\Psi(\zeta) = \sum_{k=Z_c+1}^{\infty} \zeta^k p_k$ are the partial generating functions of stable and unstable states, respectively.

To find F , one may use a simple trick [using Eq. (5)]:

$$\left(\frac{\partial^{Z_c+1+j}}{\partial \zeta^{Z_c+1+j}} \Phi(\zeta) \right) \Big|_{\zeta=0} \equiv 0, \quad j \geq 1. \quad (6a)$$

In general, this system is an infinite set of related equations. It relates all p_{Z_c+1+j} to p_0 . Together with $F(1) = 1$, it provides an *exact* solution, $F(\zeta)$, of Eq. (5). For an *Abelian* sand pile, the transition probabilities of simultaneous toppling and noise [e.g., toppling to *higher* states, $\mathbf{1} \cdot \delta^* \alpha$ (see Fig. 1)] vanish identically by definition. (Note this is the limit when noise is too *weak* to affect avalanche dynamics.) Thus, the highest achievable state is $Z_c + 2N_f$. The Markov chain is finite, and Eqs. (6a) reduce (schematically) to the system with triangular matrix (with a_{ij} and b_k being constants)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,2N_f} \\ 0 & a_{2,2} & \dots & a_{2,2N_f} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{2N_f,2N_f} \end{pmatrix} \begin{pmatrix} p_{Z_c+1} \\ p_{Z_c+2} \\ \vdots \\ p_{Z_c+2N_f} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2N_f} \end{pmatrix} p_0, \quad (6b)$$

which can be solved *exactly* and *explicitly*. In the opposite case, when noise is *not* weak, one may, however, truncate the system of Eqs. (6a) to a finite hierarchy simply by noticing that the probabilities $p_{Z_c+2N_f+j}$, $j \geq 1$, are very low since they can be reached only from *unstable* states. If one truncates at the state $Z_c + 2N_f + j$, the error in determination of all the state probabilities, p_k , will not exceed the value $(\mathbf{1} \cdot \delta^* \alpha)^j$.

Equations (6a) or (6b) constitute the (almost) exact solution, i.e., state probabilities of a site x , of a CA sand pile model in terms of state probabilities (entering through ϵ and δ) of its neighbors.

The normalization condition, $F(1) = 1$, gives the relation for the total probability for a site to be unstable:

$$\mathcal{P} \equiv \Psi(1) = \alpha[p_0 + N_f(\epsilon + 2\delta)]/2N_f. \quad (7)$$

We straightforwardly define the SO profile as the mathematical expectation value (average) of the random process:

$$\Delta h_{\text{SO}} \equiv \langle n \rangle = F'_\zeta(1), \quad (8)$$

where two unknowns $p_0(\epsilon, \delta)$ and $\Psi'_\zeta(1)$ appear and are to be found from Eqs. (6a) or (6b). For arbitrary Z_c and N_f (especially for large Z_c and N_f , i.e., in the continuous limit) the result can be easily found numerically. To obtain an analytically tractable expression, we make additional (though natural) approximations.

(1) Let us consider an asymmetric random walk on a finite chain with transition probabilities to the right and to the left, g and r , respectively. One can easily show [recursively from Eq. (2)] that $p_0(g/r) = (g/r - 1)[(g/r)^c - 1]^{-1}$, where c is a constant that is found from an expansion of $p_0(g/r)$ near $g/r \sim 1$ to give $p_0|_{g/r=1} = 1/c$. By analogy with an asymmetric random walk, we write $g/r = (\alpha + N_f\epsilon^* + 2N_f\delta^*)/\beta = 1 + N_f\epsilon + 2N_f\delta$ and

$$p_0 \approx N_f(\epsilon + 2\delta) / \{ [1 + N_f(\epsilon + 2\delta)]^{1/p_0^{(0)} - 1} \}, \quad (9)$$

where $p_0^{(0)} = p_0|_{\epsilon=\delta=0} = (Z_c - N_f + 3/2)^{-1}$ [the last follows from Eq. (5) for $\epsilon = \delta = 0$].

(2) To define $\Psi'_\zeta(1)$, we consider two limits for which $\Psi(\zeta)$ is known *a priori*. When $\epsilon = \delta = 0$, only one-step transitions (noise) exist. Therefore, from the definition, we have $\Psi'_\zeta(1)|_{\epsilon=\delta=0} = (Z_c + 1)p_{Z_c+1}$, i.e., only the first unstable state can be achieved. For ϵ, δ is sufficiently large, the states $k \in [Z_c + 1, Z_c + 2N_f]$ are roughly uniformly populated while higher states $Z_c + 2N_f + k$, $k \geq 1$ have low probability, as they can be reached only from *unstable* states. Thus, we can write $\Psi'_\zeta(1) \approx \Psi(1)[Z_c + 1 + (2N_f - 1)/2]$. From comparison of the last two equations, we conclude

$$\Psi'_\zeta(1) \approx \mathcal{P}[(Z_c + 1) + (N_f - 1/2)f(\epsilon, \delta)], \quad (10)$$

where $f = 1$ for large ϵ, δ and vanishes for $\epsilon = \delta = 0$. Because the quantity $\epsilon + 2\delta$ is a ‘‘measure of asymmetry’’ of a random walk, we choose $f(\epsilon, \delta) = 2(\epsilon + 2\delta)/[1 + (\epsilon + 2\delta)]$.

Finally, the SO local slope (for $\alpha \ll 1$) reads

$$\Delta h_{\text{SO}} \approx [p_0 + N_f(\epsilon + 2\delta)] \left(\frac{N_f - 1/2}{1 + N_f(\epsilon + 2\delta)} + \alpha \right) + (Z_c + 1 - 3N_f/2) + \delta N_f / (\epsilon + 2\delta). \quad (11)$$

Here p_0 is given by Eq. (9). Equation (11) depends on the noise strength α as well as on the toppling probabilities of adjacent sites of the pile.

One-dimensional pile. Equation (11) defines the average SO slope for every site x . The quantities ϵ and δ are defined by toppling probabilities of neighboring sites. Each site x topples with probability $\mathcal{P} = \mathcal{P}(x)$. This probability varies from site to site. In the mean-field approximation, by definition

$$\begin{aligned} \epsilon\alpha &= \mathcal{P}(x-1)[1 - \mathcal{P}(x+1)] \\ &+ \mathcal{P}(x+1)[1 - \mathcal{P}(x-1)], \end{aligned} \quad (12)$$

$$\delta\alpha = \mathcal{P}(x-1)\mathcal{P}(x+1).$$

Note, the stronger the noise, the better this ansatz works, because decorrelation of $\mathcal{P}(x \pm 1)$ is caused by noise. Equation (7) can be written as a recurrence equation for probabilities $\mathcal{P}(x)$

$$2\mathcal{P}(x) = \alpha p_0(x)/N_f + [\mathcal{P}(x-1) + \mathcal{P}(x+1)], \quad (13)$$

where $p_0(x)$ is also a function \mathcal{P} and given by Eq. (9). This equation can be solved numerically with the condition at the open (left) boundary that “influx”=“outflux.” The initial value is thus $\mathcal{P}(0) = p_{\text{sand}}(0)/N_f$. Equation (11) together with Eqs. (12) defines a spatial profile of the SO slope of the sand pile. In the continuous limit (vanishing cell size of a Markov chain), Eq.(13) is equivalent to $\mathcal{P}'_{xx} = \alpha p_0/N_f$. Thus the approximate solution matching the boundary condition is

$$\mathcal{P}(x) \approx (p_{\text{sand}} p_0 / N_f) x^2 + [p_{\text{sand}}(1 - p_0) / N_f] x. \quad (14)$$

The SO gradient profiles are shown in Fig. 2 for $Z_c = 8$, $N_f = 3$, and three values of noise strength $\alpha \approx p_{\text{sand}}$: $\alpha = 1/5000$ (low noise), $\alpha = 1/1500$, and $\alpha = 1/500$ (high noise). The average gradient profiles always have a region of relatively small gradient, “boundary layer,” near the top of the pile. This region appears due to the effect of the open (in gradient space) boundary at $x=0$. In the case of low noise, the SO gradient is almost constant throughout the pile and always below the marginally stable value. In the “overdriven” regime (i.e., high noise), a region of a *supercritical* gradient appears near the bottom of the pile. In this case the sand influx is so large that a nearly constant flow of sand

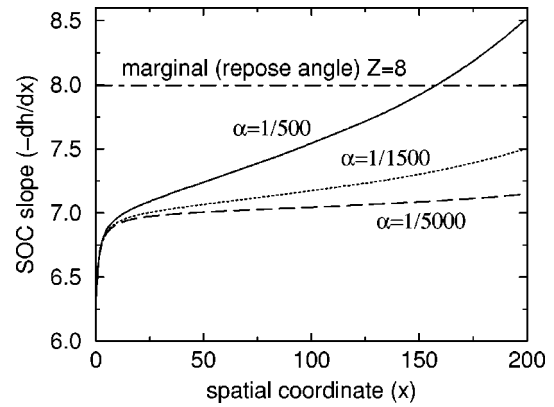


FIG. 2. The SO profiles of gradient of the pile ($Z_c=8$, $N_f=3$) for three noise levels.

forms near the bottom, thus maintaining an unstable, supercritical gradient. These results agree well with simulations [10].

In this paper, we show that the Abelian property is not necessary for an (almost) exact solvability of a sand pile CA. As an example, a spatial profile of a one-dimensional sand pile is calculated.

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